

Grafting Real Complex Projective Structures with Schottky Holonomy

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Abstract

Let $\mathcal{G}(S, \rho)$ be a graph whose vertices are complex projective structures with holonomy ρ and whose edges are graftings from one vertex to another. If ρ is quasi-Fuchsian, a theorem of Goldman implies that $\mathcal{G}(S, \rho)$ is contractible. If ρ is a quasi-Fuchsian Schottky group Baba has shown that $\mathcal{G}(S, \rho)$ is connected. We show that if ρ is a quasi-Fuchsian Schottky group $\pi_1(\mathcal{G}(S, \rho))$ is not finitely generated and there are an infinite number of (standard) projective structures which can be grafted to a common structure.

1 Introduction

A (complex) projective structure, or $(PSL_2(\widehat{\mathbb{C}}), \mathbb{C})$ -structure on an orientable surface S is a generalization of a hyperbolic structure on S and may be thought of as a Riemann structure on S with the added notion of circles. A projective structure defines a *holonomy representation* ρ via a *developing map* D and the pair (D, ρ) uniquely determines the structure. It is natural to ask to what degree the holonomy representation characterizes a projective structure. Indeed, Hejhal showed [9] that the holonomy map $\mathcal{H} : P(S) \rightarrow R(S)$ where $R(S)$ is the set of representations of the fundamental group in $PSL_2(\mathbb{C})$ up to conjugacy and $P(S)$ is the space of all projective structures on S is a local homeomorphism. Later Earle [4] and Hubbard [10] independently showed that the \mathcal{H} is holomorphic. Also Gallo, Kapovich and Marden [6] show that almost all representations of $\pi_1(S)$ into $PSL_2(\mathbb{C})$ are holonomy representations.

With almost every representation of the fundamental group of a surface being a holonomy representation of a projective structure we study to what degree the holonomy determines the projective structure. Even when the representation is discrete and faithful there are many different projective structures with that representation as its holonomy.

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Examples of this sort are constructed using a cut-and-paste operation called *grafting* which glues a projective structure on an annulus to a projective structure Σ on S . This is done by first splitting S open along a suitable curve and attaching the annulus to S along its boundary. There are many suitable curves for grafting, and this gives rise to many different projective structures. A key feature of grafting is that the result is again a projective structure on S with the same holonomy as Σ .

A standard projective structure is one whose developing map is a covering map onto its image, or equivalently, one that cannot be realized as a graft of another. Two projective structures are said to differ by an elementary move if they differ by a power of a Dehn twist about a curve with trivial holonomy. Goldman [7] has shown that any projective structures with faithful Fuchsian holonomy can be obtained by grafting one of the two standard Fuchsian projective structures along a unique set of admissible curves. Recently Baba [1] has proved an analogous result for the case that the holonomy is onto a Schottky group.

Theorem 1.1 (Baba) *Every projective structure on S with Schottky holonomy ρ is obtained by grafting a standard structure once along a multiloop on S .*

This theorem implies that $\mathcal{G}(S, \rho)$ is connected if $\rho(\pi_1(S))$ is a Schottky group. The lack of uniqueness in Theorem 1.1 is due to the fact that, in this case, there are many standard projective structures with the same holonomy. Our results exploit this profusion of standard projective structures. One of the main results of this paper imply that any two projective structures which differ by an elementary move can be grafted to a common structure, in an infinite number of ways.

Theorem 1.2 *If Σ and Σ' are standard real Schottky projective structures that differ by an elementary move then there are an infinite number of structures which are grafts of both.*

The result above implies that $\pi_1(\mathcal{G}(S, \rho))$ is not finitely generated. Since there are an infinite number of standard structures which differ by an elementary move there we also show the following.

Theorem 1.3 *There are an infinite number of standard real Schottky projective structures which can be grafted to a common structure.*

Real representations have several features which make this study more accessible. In particular ρ preserves \mathbb{R} and thus $D^{-1}(\mathbb{R})$ is a $\pi_1(S)$ invariant set in \tilde{S} which descends to a set of disjoint simple closed curves on S . So two standard projective structures Σ_1, Σ_2 on S differ by an *elementary move* if the real curves of Σ_1 differ from the real curves of Σ_2 by an elementary move. These so-called *real curves* characterize a projective structure with real holonomy in the following way.

Theorem 1.4 *Fix an oriented surface S and a real representation ρ . Let $\Sigma_1 = (D_1, \rho)$ and $\Sigma_2 = (D_2, \rho)$ be projective structures on S . Then*

$$D_1^{-1}(\mathbb{R}) \simeq D_2^{-1}(\mathbb{R}) \iff \Sigma_1 = \Sigma_2.$$

Let $T(S)$ denote the space of all hyperbolic structures on S and $\mathcal{ML}(S)$ the set of measured laminations on S . The representation ρ determines an $X \in T(S)$. Since ρ is real we assign a measure of π to each leaf in $D^{-1}(\mathbb{R})$. Theorem 1.4 can be viewed as a special case of the following theorem of Thurston which shows that a projective structure may be viewed as a hyperbolic surface in \mathbb{H}^3 which is bent along a geodesic lamination.

Theorem 1.5 (Thurston) *If $P(S)$ denote the set of all projective structures on S then*

$$\Theta : P(S) \rightarrow T(S) \times \mathcal{ML}(S)$$

given by θ is an isomorphism.

One application of Theorem 1.4 is a formulation of a topological proof of the following theorem of Goldman which motivated much of this work.

Theorem 1.6 (Goldman) *Let ρ be a faithful real representation and Σ_0 the standard complex projective structure with holonomy ρ . For any complex projective structure Σ with holonomy ρ , there exists a unique collection of simple closed curves \mathcal{C} in Σ_0 such that grafting Σ_0 by \mathcal{C} obtains Σ .*

Theorem 1.6 implies that $\mathcal{G}(S, \rho)$ is connected if $\rho : \pi_1 \rightarrow PSL_2(\mathbb{R})$ is injective. Theorem 1.4 uses a characterization of grafting along so-called *spiraling* curves, i.e., curves whose lifts to \tilde{S} develop to spherical spirals (loxodromes) in $\hat{\mathbb{C}}$. The real curves obtained by grafting along such curves are obtained by a surgery procedure, called \sharp - \flat operations on the union of the initial real curves and curves isotopic to the grafting curve. Such operations are used by Ito 1.7 to prove the following.

Theorem 1.7 (Ito) *Let $Q(S)$ be the subset of $P(S)$ of projective structures with quasi-Fuchsian holonomy. A non-standard component of $Q(S)$ (consisting of quasi-Fuchsian projective structures that are not standard) is not a topological manifold with boundary and any two components of $Q(S)$ have intersecting closures.*

This locally defined operation first removes intersections and then reconnects nearby curves so that the resulting curves are disjoint. Its relation to spiraling curves is expressed below.

Theorem 1.8 *If γ is an spiraling curve admissible for grafting in Σ , then the real curves of the graft of Σ along γ are obtained by either a \sharp or \flat operation on the union of the real curves of Σ with two curves isotopic to γ .*

1.1 Idea of Proof of Theorems 1.2 and 1.3

By Theorem 1.4 complex projective structures with real holonomy are classified by the homotopy classes of their real curves. Theorem 1.8 shows that real curves of certain (spiral) grafted structures are obtained from the original real curves using surgery operations. We will show that the \sharp and \flat operations on spiraling curves in S reduce to \sharp and \flat operations on curves in a torus. In the torus these operations are closely related to Dehn twists, and this implies that the real curves produced by grafting can be constructed by Dehn twists of the initial real curves about a meridian in S which intersects the real curves exactly twice. This technical condition on β makes it easy to see the relationship to spiraling curves. Powers of Dehn twists correspond to higher frequencies of spiraling. The meridian is used to identify other projective structures, via an elementary move, and we will show that certain grafts of these other structures are also grafts of the initial structures.

For any fixed real lamination λ of a standard structure Σ and meridian β which intersects λ exactly twice, we will show that the each power of a Dehn twist of λ about β are the real curves of some standard structure Σ' . We will then use the Dehn twist to produce admissible curves for Σ and Σ' so that the graftings of both coincide.

2 Preliminaries

A *complex projective structure* Σ on an orientable surface S is a maximal atlas $(\phi_i : U_i \rightarrow \widehat{\mathbb{C}})_{i \in I}$ where ϕ_i is a homeomorphism onto its image and the transition functions are restrictions of elements in $PSL_2(\widehat{\mathbb{C}})$, to connected components. Since a maximal atlas is unique the set of charts giving a projective structure is unique.

A global definition of a projective structure is an equivalence class of developing pairs $[(D, \rho)]$ where the *developing map* $D : \tilde{S} \rightarrow \widehat{\mathbb{C}}$ globalizes the charts in the atlas and the homomorphism $\rho : \pi_1(S) \rightarrow PSL_2(\widehat{\mathbb{C}})$ globalizes the coordinate changes. Then ρ is a *holonomy* representation and commutes with D

$$D(\gamma x) = \rho(\gamma)D(x)$$

for all $\gamma \in \pi_1(S)$ and $x \in \tilde{S}$. The developing map D is unique up to pre-composition with a map that is a lift of map of S that is isotopic to the identity and post-composition with $PSL_2(\widehat{\mathbb{C}})$ and ρ is unique up to conjugation. To ease notation we let (D, ρ) denote a projective structure where D is a representative developing map and ρ is a representative holonomy representation of the equivalence class $[(D, \rho)]$.

A basic example of a complex projective structure on S is the quotient of the upper half plane in \mathbb{C} by a Fuchsian group, i.e., a discrete subgroup of $PSL_2(\mathbb{R})$ isomorphic to $\pi_1(S)$. The quotient is both a hyperbolic structure and a complex projective structure. Indeed, since $\text{Isom}(\mathbb{H}^2) \cong PSL_2(\mathbb{R}) \subset PSL_2(\mathbb{C})$ and $\mathbb{H}^2 \subset \mathbb{C}$ every hyperbolic structure determines a complex projective structure. More

generally, if Γ is a Kleinian group, (a finitely generated discrete subgroup of $PSL_2(\widehat{\mathbb{C}})$) isomorphic to $\pi_1(S)$ and Ω is a connected component of the domain of discontinuity of Γ which is fixed by Γ then the quotient Ω/Γ is a complex projective structure on S . However, as we shall see developing maps are not necessarily injective and holonomy representations are not necessarily discrete or faithful.

In the next section we first formulate a precise global definition of a projective structure, expressed a developing map and holonomy representation. After mentioning relevant earlier work in this area, we describe the grafting operation in detail, a construction which is independent of the underlying holonomy representation. Restricting our discussion to projective structures whose holonomy representations are real, we then discuss the effect of grafting such structures.

2.1 The Developing Map

We first give a precise definition of the developing map. Let $\mathcal{A} = (\phi_i, U_i)$ be a maximal atlas defining a projective structure Σ on an oriented surface S . Choose a point x in S and assume $x \in U_0$. A lift \tilde{x} of x is contained in a lift of \tilde{U}_0 of U_0 and a chart (ϕ_0, U_0) on S lifts to a chart $(\tilde{\phi}_0, \tilde{U}_0)$ on \tilde{S} . Since \tilde{S} is simply connected $(\tilde{\phi}_0, \tilde{U}_0)$ extends to a chart D_0 on all of \tilde{S} . This chart is maximal since its domain is all of \tilde{S} .

This global chart is unique in the sense that if two global charts agree on an open set of \tilde{S} then there is an isotopy of \tilde{S} , after which they agree on all of \tilde{S} . Therefore, this global chart is unique up to an isotopy of \tilde{S} .

For another lift \tilde{x}_g of x , there is some g in $\pi_1(S)$ such that $g\tilde{x} = \tilde{x}_g$ and $g\tilde{U}_0 = \tilde{U}_g$ is a lift of U_0 . There is a lift $(\tilde{\phi}_g, \tilde{U}_g)$ of (ϕ_0, U_0) which extends to a maximal global chart D_g on \tilde{S} . Since D_0 and D_g arise from lifts of U_0 they differ by pre-composing with an element of the fundamental group and so there is some g such that

$$D_0 = D_g \circ g.$$

Both D_g and D_0 are global charts on \tilde{S} and so there is a transition map ϕ_g which we post-compose with D_0 so that the composition agrees with D_g on all of \tilde{S} , i.e.,

$$D_0 = \phi_g \circ D_g.$$

We call each D_i a *developing map*. This determines a map $\rho : \pi_1(S) \rightarrow PSL_2(\mathbb{C})$ given by

$$\rho(g) = \phi_g.$$

We show that ρ is a homomorphism. If g_1 and g_2 are elements of $\pi_1(S)$, then $D_0 \circ g_1 \circ g_2$ is a global chart on \tilde{S} . By the definition of an atlas there is a Mobius transformation ϕ such that

$$\phi \circ D_0 = D_0 \circ g_1 \circ g_2.$$

Similarly, there are Mobius transformations ϕ_1 and ϕ_2 such that

$$\phi \circ D_0 = \phi_2 \circ D_0 \circ g_1 = \phi_1 \circ \phi_2 \circ D_0.$$

Therefore $\phi = \phi_1 \circ \phi_2$ and thus ρ is a homomorphism we call the *holonomy representation*.

If another point y in S is chosen, the resulting developing map differs from D_0 by a Möbius transformation since both are global charts on \tilde{S} . Likewise the resulting holonomy representation differs from the initial one by conjugation in $PSL_2(\mathbb{C})$.

Fix a projective structure $\Sigma(D, \rho)$ on S . If X is a subset of S , and \tilde{X} is a lift of X to the universal cover, then we say $\tilde{X} = D(\tilde{X})$ is the developed image of X .

2.2 Grafting

The process of grafting is defined for an admissible curve γ in S , and a projective structure $\Sigma = (D, \rho)$.

Definition 2.1 *A simple closed curve γ in S is admissible if :*

- $\rho(\gamma)$ is hyperbolic and
- $D|_{\tilde{\gamma}}$ is an embedding of each lift $\tilde{\gamma}$ of γ

where $\rho(\gamma)$ denotes the image under ρ of an element of $\pi_1(S)$ in the free homotopy class of γ . We say the homotopy class $[\gamma]$ of γ has an admissible representative if there is an admissible curve in $[\gamma]$.

2.3 Hopf Torus

To every grafting there is an associated torus, which we now briefly discuss. Since γ is admissible $\rho(\gamma)$ is a hyperbolic element in $PSL_2(\mathbb{C})$. There are two fixed points $Fix(\rho(\gamma))$ of $\langle \rho(\gamma) \rangle$. The quotient $T = \hat{\mathbb{C}} - Fix(\rho(\gamma))$ under the action of $\langle \rho(\gamma) \rangle$ is a projective structure on a torus we call the Hopf torus. Each lift of γ has a corresponding Hopf torus. The developing maps of these Hopf tori differ by post-composition with a Möbius transformation and their holonomy representations differ by conjugation in $PSL_2(\mathbb{C})$.

2.4 The Collapsing Map

Let $N(\gamma)$ denote an annular neighborhood of an admissible curve $\gamma \in S$ and let $\eta : S^1 \times (0, 1) \rightarrow N(\gamma)$ be a homeomorphism such that $\eta(S^1, \frac{1}{2}) = \gamma$. We define a *collapsing* map $\nu : S \rightarrow S$ which is the identity on $S \setminus N(\gamma)$, collapses $\eta(\gamma \times (\frac{1}{4}, \frac{3}{4}))$ to γ and expands $\eta(\gamma \times (0, \frac{1}{4}))$ and $\eta(\gamma \times (\frac{3}{4}, 1))$ as follows and as shown in Figure 1.

$$\nu(y) = \begin{cases} y & \text{if } y \in S \setminus N(\gamma) \\ \eta(s, \frac{1}{2}) & \text{if } y = \eta(s, t) \text{ for } t \in (\frac{1}{4}, \frac{3}{4}) \\ \eta(s, 2t) & \text{if } y = \eta(s, t) \text{ for } t \in (0, \frac{1}{4}) \\ \eta(s, 2t - 1) & \text{if } y = \eta(s, t) \text{ for } t \in (\frac{3}{4}, 1) \end{cases}$$

Choose a lift to the universal cover $\tilde{\nu} : \tilde{S} \rightarrow \tilde{S}$ which is homotopic to the identity outside of the preimage of $N(\gamma)$.



Figure 1: The collapsing map

2.5 Projective Structure on Annulus

Choose a lift $\tilde{\gamma}$ of γ in \tilde{S} . Let $A = \eta(\gamma \times (\frac{1}{4}, \frac{3}{4}))$ and \tilde{A} be the lift to \tilde{S} containing $\tilde{\gamma}$. Since γ is admissible $D(\tilde{\gamma})$ is a simple arc in $\widehat{\mathbb{C}}$ with distinct endpoints p_1 and p_2 .

For $x \in S^1$ let $a_x = \eta(x \times (\frac{1}{4}, \frac{3}{4}))$. There is a unique non-trivial homotopy class of simple closed curves in $\widehat{\mathbb{C}} - \text{Fix}(\rho(\gamma))$, and we call this class $[c]$. Let h denote an orientation preserving map to the torus T $h : A \rightarrow T$ which has a lift $\tilde{h} : \tilde{A} \rightarrow \widehat{\mathbb{C}} - \text{Fix}(\rho(\gamma))$ which is surjective, injective away from $\tilde{\gamma}$, on each component of $\partial(\tilde{A})$ restricts to

$$\tilde{h} = D \circ \tilde{\nu}$$

and for each $x \in S^1$ has the property that

$$\tilde{h}(a_x) = \delta_x$$

where $\delta_x \in [c]$. These properties ensure that arcs of A that are collapsed to single points of γ develop to simple closed curves in $\widehat{\mathbb{C}} - \text{Fix}(\rho(\gamma))$.

Since \tilde{h} is a lift of h , \tilde{h} is equivariant with respect to the action of $\langle a \rangle$ the \mathbb{Z} -subgroup that fixes $\tilde{\gamma}$. That is, for every $x \in \tilde{\gamma}$

$$\tilde{h}(ax) = \rho(a)\tilde{h}(x).$$

Assume x lies in $\pi^{-1}(A)$ where $\pi : \tilde{S} \rightarrow S$ is the universal covering. Then there is some non-trivial $g_1 \in \pi_1(S)$ for which g_1x lies in \tilde{A} . We extend \tilde{h} to $\pi^{-1}(A)$ by

$$\tilde{h}(x) = \rho(g_1^{-1})\tilde{h}(g_1x) \text{ where } x \in \pi^{-1}(A), g_1x \in \tilde{A} \text{ and } g_1 \in \pi_1(S).$$

Note that there are many g such that $gx \in \tilde{A}$. If $g_2x \in \tilde{A}$ note that $g_1g_2^{-1}$ fixes \tilde{A} . Then

$$\rho(g_2^{-1})\tilde{h}(g_2x) = \rho(g_2^{-1})\rho(g_2g_1^{-1})\tilde{h}(g_1g_2^{-1}g_2x) = \rho(g_1^{-1})\tilde{h}(g_1x).$$

Therefore the definition of \tilde{h} is independent of the choice of g .

For any $x \in \pi^{-1}(A)$, and $g \in \pi_1(S)$ we have

$$\tilde{h}(gx) = \rho(g_1^{-1})\tilde{h}(g_1gx) = \rho(g_1^{-1}g_1)\tilde{h}(gx) = \rho(g)\tilde{h}(x).$$

Then \tilde{h} is equivariant with respect to the action of the fundamental group of S .

2.6 Grafted Developing Map

Given a projective structure on S and the collapsing map ν we define a new developing map $D' : \tilde{S} \rightarrow \hat{\mathbb{C}}$ in the following way. For \tilde{A} a lift of A to \tilde{S} we set

$$D'(x) = \begin{cases} D \circ \tilde{\nu}(x) & \text{if } x \in \tilde{S} \setminus \pi^{-1}(A) \\ \tilde{h}(x) & \text{if } x \in \pi^{-1}(A) \end{cases}$$

Since ν is a homotopy equivalence we have the commutativity relation $\tilde{\nu}(gx) = g\tilde{\nu}(x)$ for all $g \in \pi_1(S)$. For $g \in \pi_1(S)$, gx lies in $\pi^{-1}(A)$ if and only if x lies in $\pi^{-1}(S)$ so we have

$$D'(gx) = \begin{cases} D \circ \tilde{\nu}(gx) = D(g\tilde{\nu}(x)) = \rho(g)D'(x) & \text{if } gx \in \tilde{S} \setminus \pi^{-1}(A) \\ \tilde{h}(gx) = \rho(g)\tilde{h}(x) = \rho(g)D'(x) & \text{if } gx \in \pi^{-1}(A) \end{cases}$$

which shows that D' is π_1 -equivariant with holonomy ρ . This defines a new projective structure $(D', \rho) = Gr_\gamma(\Sigma)$ on S .

3 Real Holonomy

This section consists of facts about complex projective structures with real holonomy. In the first section no injectivity assumption is made on the holonomy representation. Here we reduce the classification of projective structures with real holonomy to the classification of the homotopy class of their so called *real curves*. By the *real curves* of (D, ρ) we refer to all curves in the surface whose lift to the universal cover lies in the preimage of \mathbb{R} under D . This reduction is used to construct an alternate proof a Theorem of Goldman (see [7]).

3.1 Real Curves of a Projective Structure

In the remainder of this section, we will assume the representation ρ is real. To ease notation we make the following convention.

Notation 3.1 *Let $\Sigma = (D, \rho)$ be a projective structure on S and let $\pi : \tilde{S} \rightarrow S$ be the universal covering projection. Set*

$$\lambda = \pi(D^{-1}(\mathbb{R}))$$

and let $\tilde{\Sigma}$ denote the universal cover of the projective structure Σ .

Lemma 3.2 *Let (D, ρ) denote a projective structure on S , and $\pi : \tilde{S} \rightarrow S$ the universal covering. Then λ is a finite collection of disjoint closed curves on S .*

Proof :

Since D is continuous and $\mathbb{R} \cup \infty$ is closed in $\hat{\mathbb{C}}$, it follows that $\tilde{\lambda} = D^{-1}(\mathbb{R} \cup \infty)$ is closed in \tilde{S} . The real representation ρ preserves $\mathbb{R} \cup \infty$ and so $\tilde{\lambda}$ is a closed π_1 -invariant set in \tilde{S} . Then λ is a closed set in the compact surface S , and is

therefore compact. Each point in λ in S has a neighborhood homeomorphic to the the unit disk in \mathbb{C} with the real line corresponding to the real curve since D is a local homeomorphism. Therefore each connected component of the real curves is a 1-manifold. A compact 1-manifold is homeomorphic to a circle so each connected component of λ is a closed curve. Since λ is compact the number of curves in λ is finite. \square

A geometric disk of $\widehat{\mathbb{C}}$ is an open subset of $\widehat{\mathbb{C}}$ whose boundary is a geometric circle. A geometric disk of \tilde{S} is an open subset U such that $D : U \rightarrow D(U)$ is a homeomorphism onto a geometric disk of $\widehat{\mathbb{C}}$. The set of geometric disks of \tilde{S} is partially ordered by inclusions. We call a maximal geometric disk a maximal disk.

Lemma 3.3 *Every point of \tilde{S} lies in a proper maximal disk.*

Next, we note that the real curves of a projective structure are closed curves on the surface. Let $D^2 = \mathbb{H}^2 \cup S^1$ be the compactification of \tilde{H}^2 . If U is a maximal disk of \tilde{S} we call

$$U_\infty = \overline{U} - \tilde{S}$$

the ideal set of U where \overline{U} is the closure of the compactification of U in \tilde{S} . Since \overline{U} is conformally equivalent to a closed ball we may form $C(U_\infty)$ the convex hull of U_∞ in U . Then $C(U_\infty)$ is the smallest convex set in \overline{U} containing U_∞ .

Lemma 3.4 *Every point of \tilde{S} lies the convex hull $C(U_\infty)$ of a unique maximal disk and $\partial(U_\infty)$ is closed in \tilde{S} .*

Let $D^3 = \mathbb{H}^3 \cup S^2$ be the compactification of \mathbb{H}^3 , $\mathcal{H}(D(U))$ denote the hyperbolic plane in \mathbb{H}^3 that contains the boundary of $D(U)$. For each U there is a nearest point retraction $\Phi_U : D(U) \rightarrow \mathbb{H}^2 \subset \mathbb{H}^3$ given by

$$\Phi_U : D(U) \rightarrow \mathcal{H}(D(U)).$$

By the uniqueness lemma for every $x \in C(U_\infty)$ there is a map $\Psi : \tilde{S} \rightarrow \mathbb{H}^3$ given by

$$\Psi(x) = \Phi_U(D(x)).$$

In general the map Ψ is not locally injective. Denote by \mathcal{B} the set of those $C(U_\infty^i)$ on which Ψ is not injective. There is a lift of a homotopy equivalence $\tilde{\nu}$ of \tilde{S} which collapses each component of \mathcal{B} to a common arc. Then

$$\Psi : \tilde{\nu}(S) \rightarrow \mathbb{H}^2 \subset \mathbb{H}^3$$

is injective and pulls back a hyperbolic metric on $\tilde{\nu}(S)$. Since $\tilde{\nu}$ is a homotopy equivalence this induces a hyperbolic metric σ on S .

It can be seen that Ψ is a *pleated* map, that is a continuous map from a hyperbolic surface S into a hyperbolic 3-manifold (\mathbb{H}^3 in this case) such that for any point $x \in S$, there is a geodesic in S containing x which is mapped to a geodesic in the 3-manifold, and the path metric induced from \mathbb{H}^3 by Ψ coincides with the hyperbolic metric on S . The image of Ψ is a pleated surface in \mathbb{H}^3 which is bent along a lamination λ (the pleating locus) in the sense that $\Psi(S)|\lambda$ is totally geodesic. The geodesic lamination λ supports a measure, if Ψ is locally convex and this measure corresponds to the bending angle.

3.2 The Measured Lamination

Here we realize the real curves as a measured lamination with isolated leaves, each of weight $m\pi$ for $m \in \mathbb{Z}$.

Since we assume ρ is real, the limit set of Γ is contained in $\mathbb{R} \cup \infty$, and the convex hull C of the limit set of Λ in \mathbb{H}^3 is contained in the equatorial half plane \mathbf{H} . The *pleating locus* of \tilde{S} is the set of points L in \tilde{S} through which there passes one and only one open geodesic arc a for which $\Psi(a)$ is a geodesic in $\Psi(\tilde{S})$. It is shown in [14] p. 177, that the image of the pleated map, a pleated surface S' , is contained in convex hull $\mathcal{CH}(\Gamma)$ of Γ and that L is a measured geodesic lamination on \tilde{S} .

Then, following [15], L assigns to any smooth compact arc c on \tilde{S} intersecting L transversely, and whose endpoints lie in the complement of L a finite Borel measure μ on c with support in $c \cap L$. By Lemma 3.3 each endpoint e_i of c is contained in the convex hull $C(U_{e_i})$ of unique maximal disk U_{e_i} . Since the e_i do not lie in the pleating locus there are at least two geodesics of U_{e_i} through e_i which are mapped to geodesics by Ψ . Now each U_{e_i} contains more than two ideal points and the $C(U_{e_i})$ are two-dimensional. It follows that $\Psi(C(U_{e_i})) \subset \mathbf{H}$ where \mathbf{H} denotes the equatorial half plane in \mathbb{H}^3 .

Let $\Theta(s, t)$ denote the dihedral angle of intersection of the circles U_s and U_t . In [12] it is shown that $\mu(c) = \varphi'$ with

$$\varphi = \lim \sum \Theta(0, t_1) + \dots + \Theta(t_n, t)$$

where the sum runs over subdivisions of $[0, t] = c$ and the limit is taken as the width of the widest interval goes to zero.

If $\mu(c) < \pi$ then $\Theta(D(U_{e_1}), D(U_{e_2})) \leq \lim(\Theta(0, t_1) + \dots + \Theta(t_n, t)) < \pi$. Then $\mu(c) = 0$, since otherwise $\Psi(C(U_{e_i})) \subset \mathbf{H}$ implies a contradiction. It follows easily that if $\mu(c) = m\pi$ for $m \in \mathbb{Z}^+$ then c is an atom of μ and all atoms have measure $m\pi$ and correspond to separated leaves of the lamination. If $\mu(c) > \pi$ and has no atoms then there is a $c' \subset c$ such that $\mu(c') < \pi$ and we conclude that the support for μ consists of isolated leaves each with weight $m\pi$.

Proposition 3.5 *Fix a projective structure (D, ρ) on S where ρ is a discrete real representation onto Γ . Let U denote a connected component of $U \subset \tilde{S} \setminus D^{-1}(\mathbb{R})$. Then U is a maximal disk of \tilde{S} which is mapped either onto the upper or lower half plane by D .*

Proof :

Fix U a connected component of $\tilde{S} \setminus D^{-1}(\mathbb{R})$. By Lemma 3.3 for each $x \in \tilde{S}$ there is a unique maximal disk V_x such that x lies in $C(V_x)$. By construction, $D(V_x)$ is a round disk in $\hat{\mathbb{C}}$. The complete hyperbolic metric on $D(V_x)$ pulls back to a complete hyperbolic metric on V_x . Let H_x denote the plane in \mathbb{H}^3 whose boundary is the same as the boundary of $D(V_x)$.

Since V_x is maximal, $C(V_x)$ has at least two ideal points which bound a geodesic g_x in the hyperbolic metric on V_x containing x . As argued above, ρ

being real implies that $\Psi(C(V_x)) \subset \mathbf{H}$. The limit points of V_x are mapped into \mathbb{R} by D , and thus H_x intersects \mathbf{H} . In fact, since $\Psi(\tilde{S}) \subset \mathbf{H}$ we have

$$\Psi(x) \in H_x \cap \mathbf{H}.$$

If $H_x = \mathbf{H}$ then $V_x = U$ and the proof is complete, otherwise $H_x \cap \mathbf{H}$ is a geodesic.

Since $x \in U$, without loss of generality we can assume $D(x) \in \Omega_+$. Let g^\perp be the ray based at $\Psi(x)$ that is perpendicular to \mathbf{H} and has ideal endpoint p in Ω_+ . The ray from $\Psi(x)$ to $D(x)$ will make an acute angle with g^\perp so p will be in $D(V_x)$.

Let $y = D^{-1}(p) \cap V_x$, then $y \in U$ and to complete the proof we show that $V_y = U$. As before let H_y be the plane in \mathbb{H}^3 whose boundary is $\partial(D(V_y))$. Since the plane H_y divides \mathbb{H}^3 into two half spaces, one of these half spaces C has $D(V_y)$ as its ideal boundary and contains p .

Since $\Psi(\tilde{S}) \subset \mathbf{H}$ the geodesic in \mathbb{H}^3 with endpoint p that is orthogonal to H_y intersects H_y in \mathbf{H} . Note that the horosphere based at p that is tangent to \mathbb{H} intersects \mathbb{H} at $\Psi(x)$. Therefore any horoball based at p that intersects \mathbb{H} will contain $\Psi(x)$. Since the horoball based at p that is tangent H_y intersects H_y at a point in \mathbb{H} this horoball contains $\Psi(x)$. As this horoball will be contained in C we have that $\Psi(x) \in C$ and the geodesic $\Psi(g_x)$ which contains $\Psi(x)$ will also intersect C . Therefore one of these endpoints must be contained in $D(V_y)$ the ideal boundary of C .

We will show that D is injective on $V_x \cup V_y$. First note that since D is a local homeomorphism it is an open map and $D(V_x \cap V_y)$ is an open subset of $D(V_x) \cap D(V_y)$. The continuity of D implies that it is also a closed subset of $D(V_x) \cap D(V_y)$ and since the $V_x \cap V_y \neq \emptyset$ it is also not empty. This implies that $D(V_x \cap V_y) = D(V_x) \cap D(V_y)$.

Since D is injective on both V_x and V_y to show that it is locally injective on their union we need to show that if $q_0 \in V_x \setminus V_y$ and $q_1 \in V_y \setminus V_x$ then $D(q_0) \neq D(q_1)$. We proceed by contradiction. If $D(q_0) = D(q_1)$ then the image is in $D(V_x) \cap D(V_y)$. By the above paragraph there is then an $q_2 \in V_x \cap V_y$ such that $D(q_0) = D(q_2)$ contradicting the local injectivity of D on V_x .

Now let z_∞ be a point $D(V_x) \cap D(V_y)$ and let p_∞ be the unique point in V_y such that $D(p_\infty) = z_\infty$. There will then be a sequence of points z_i in $D(V_x) \cap D(V_y)$ such that $z_i \rightarrow z_\infty$. Let p_i be the unique point in $V_x \cup V_y$ such that $D(p_i) = z_i$. Since $D(V_x \cap V_y) = D(V_x) \cap D(V_y)$ we have that $p_i \in V_x \cap V_y$. Since D is a homeomorphism on V_y we also have $p_i \rightarrow p_\infty$. Therefore z_∞ is not an ideal point of V_x and V_y is a maximal disk contained in U where maximality then implies $V_y = U$.

□

Lemma 3.6 *Let ρ be a discrete real representation. If γ is an essential closed curve disjoint from λ in $\Sigma(\rho, \lambda)$, then γ is admissible.*

Proof : Since γ is essential, any lift of γ to \tilde{S} has two distinct endpoints on the boundary. By Proposition 3.5 any such lift is contained in a maximal disk

since γ is disjoint from λ . The developing map is injective on maximal disks, and is therefore injective on the homotopy class of γ .

Additionally, the extension of the developing map to the closure of \tilde{S} is injective on the closure of each maximal disk. The holonomy of γ is hyperbolic since S is closed, therefore the developed image of the endpoints are distinct. Thus, γ is admissible. \square

The following theorem gives a topological way to distinguish any two real complex projective structures.

Theorem 3.7 *Fix an oriented surface S and a discrete real representation*

$$\rho : \pi_1(S) \rightarrow \Gamma < PSL_2(\mathbb{R}).$$

Let $\Sigma_0 = (D_0, \rho)$ and $\Sigma_1 = (D_1, \rho)$. Then

$$\lambda_0 \simeq \lambda_1 \iff \Sigma_0 = \Sigma_1.$$

Proof : Assume $\Sigma_0 = \Sigma_1$. Then $D_0 \simeq D_1$ and there is a homeomorphism $g : S \rightarrow S$ isotopic to the identity and a lift $\tilde{g} : \tilde{S} \rightarrow \tilde{S}$ so that

$$D_0 = D_1.$$

Now suppose $x_0 \in \tilde{\lambda}_0$. Then

$$D_1 \circ \tilde{g}(x_0) = D_0(x_0) \in \mathbb{R}$$

which implies $\tilde{g}(x_0) \in \lambda_1$ and thus

$$\tilde{g}(\lambda_0) \subset \lambda_1.$$

By switching the roles of D_0 and D_1 we obtain an inverse, \tilde{g}^{-1} , for \tilde{g} . It follows that $\tilde{g}^{-1}(\lambda_1) \subset \lambda_0$, and consequently

$$\tilde{\lambda}_1 \subset \tilde{g}(\tilde{\lambda}_0)$$

which implies $\tilde{g}(\tilde{\lambda}_0) = \tilde{\lambda}_1$ and thus $g(\lambda_0) = \lambda_1$ where $g : S \rightarrow S$ is isotopic to the identity and lifts to \tilde{g} . Thus $\lambda_1 \simeq \lambda_0$. The other direction of the theorem will use the following lemmas. A proof of the first can be found in [3].

Lemma 3.8 *A homeomorphism h of S is isotopic to the identity if and only if h has a lift $\tilde{h} : \tilde{S} \rightarrow \tilde{S}$ which extends to the identity on the boundary of \tilde{S} .*

Lemma 3.9 *Let $\Sigma_0 = (D_0, \rho)$ and $\Sigma_1 = (D_1, \rho)$ be projective structures with real discrete holonomy on an oriented surface S . Suppose that an essential nonannular subsurface $X \subset S$ is disjoint from λ_0 and λ_1 . Then if \tilde{X} is a lift of X in \tilde{S} , D_0 and D_1 both map \tilde{X} into the same half plane*

Proof :

By Lemma 3.5 each developing map D_i is injective on each component of $\tilde{S} \setminus \tilde{\lambda}_i$. Since \tilde{X} is contained in a component of $\tilde{S} \setminus \tilde{\lambda}_i$, D_i is injective on \tilde{X} . Let l be a component of $\partial \tilde{X}$ and let $g \in \pi_1(S)$ be the deck transformation that fixes l . We continuously extend D_i to the endpoints p^\pm of l in the compactification of \tilde{S} by setting

$$D_i(p^\pm) = \lim_{k \rightarrow \pm\infty} \rho(g^k) D_i(x).$$

The extension of the D_i to the boundary points depends only on the holonomy, not the particular developing map so the D_i agree on the boundary points of \tilde{X} .

The orientation on S lifts to an orientation on \tilde{S} . The orientation of \tilde{S} induces an orientation of the $\partial \tilde{S}$ which induces a cyclic ordering of points in $\partial \tilde{S}$. In particular there is a cyclic order on the boundary points of \tilde{X} in $\partial \tilde{S}$. Additionally, the orientation of the Riemann sphere induces an orientation on the upper and lower half planes. The orientation of the half planes then determines a cyclic ordering of the points in $\mathbb{R} \cup \infty$, and these two orderings are opposite.

If $D_i(\tilde{X}) \subset \Omega_+ (\Omega_-)$ then the cyclic ordering of the endpoints of the image must agree with the cyclic ordering of $\mathbb{R} \cup \infty$ induced by $\Omega_+ (\Omega_-)$, since D_i has real holonomy and is orientation preserving. Since X is nonannular \tilde{X} has at least three endpoints and thus each image $D_i(\tilde{X})$ then has at least three endpoints. Reversing the cyclic order on a set with more than three points gives a different cyclic order. For these endpoints to have the same cyclic ordering both images must lie in the same half plane. Therefore D_0 and D_1 must map \tilde{X} to the same half plane. □

Lemma 3.10 *Let (D_0, ρ) and (D_1, ρ) be projective structures on S with the same real lamination λ . Then*

$$D_0(\tilde{X}) = D_1(\tilde{X})$$

where \tilde{X} is any component of $\tilde{S} \setminus \tilde{\lambda}$.

Proof : By Proposition 3.5 every component of $\tilde{S} \setminus \tilde{\lambda}$ is mapped onto either the upper or lower half plane. If X is nonannular the Lemma follows from Lemma 3.9. Since S is closed and has genus greater than 2 there is some nonannular subsurface $Y \subset S \setminus \lambda$. Let X be a possibly nonannular component of $S \setminus \lambda$. By Lemma 3.2 there are a finite number of components $X_j \subset S \setminus \lambda$ with $j \in [0, n]$ such that $X_0 = Y$, and X_i and X_{i+1} are adjacent.

We now proceed by induction on the length of the sequence. We have already observed that $D_0(\tilde{Y}) = D_1(\tilde{Y})$ for all preimages \tilde{Y} of Y . This takes care of the base case when $n = 0$. Now assume that the lemma holds for all components that can be connected to Y by a sequence of length $\leq n - 1$ and we will prove that it holds for sequences of length n . If \tilde{X} is a component of the preimage of $X = X_n$ then there is a component \tilde{X}_{n-1} of the preimage of X_{n-1} that is adjacent to \tilde{X} . Since D_i is a local homeomorphism and $\tilde{\lambda}$ is mapped to $R \cup \infty$, adjacent

components are mapped to opposite half planes. Since $D_0(\tilde{X}_{n-1}) = D_1(\tilde{X}_{n-1})$ we have $D_0(\tilde{X}) = D_1(\tilde{X})$ completing the induction step and the proof.

Lemma 3.11 *Let (D_0, ρ) and (D_1, ρ) be projective structures on S with the same real lamination λ . Then there is a $\pi_1(S)$ -equivariant map $\tilde{f} : \tilde{S} \rightarrow \tilde{S}$ which is a lift of a map f that is isotopic to the identity such that*

$$D_0 = D_1 \circ \tilde{f}.$$

Proof :

We first construct the map \tilde{f} on $\tilde{\lambda}$. This requires that D_i for $i \in \{0, 1\}$ be injective on components of $\tilde{\lambda}$, and we show this now. Let $\tilde{\gamma}$ be a component of $\tilde{\lambda}$ and let \tilde{X} be a component of $\tilde{S} \setminus \tilde{\lambda}$ such that $\tilde{\gamma}$ is in $\partial \tilde{X}$.

Let $\{p_1, p_2\}$ denote two points of $\tilde{\gamma}$ so that $D_i(p_1) = D_i(p_2)$. Since D_i is a local homeomorphism there are open neighborhoods U_1 and U_2 of p_1 and p_2 and a geometric disk V in $\hat{\mathbb{C}}$ with center at $D_i(p_1) = D_i(p_2)$ for which D_i is a homeomorphism of each U_i onto V . By Lemma 3.10 we may assume $D_i(\tilde{X}) = \Omega_+$ and we let $\{z_k\}$ be a sequence of points in $\Omega_+ \cap V$ which converge to $D_i(p_1) = D_i(p_2)$.

Let $x_k = D_i^{-1}(z_k) \cap U_1$ and $y_k = D_i^{-1}(z_k) \cap U_2$. Since D_i is a local homeomorphism D_i^{-1} is continuous so $z_k \in \Omega_+$ implies that x_k and y_k are contained in \tilde{X} . Also, since $z_k \rightarrow D(p_1) = D(p_2)$ the continuity of D_i^{-1} implies that $x_k \rightarrow p_1$ and $y_k \rightarrow p_2$. But Lemma 3.5 implies that \tilde{X} is a maximal disk, so D_i is injective on \tilde{X} and therefore $p_1 = p_2$. This implies that D_i is injective on components of $\tilde{\lambda}$.

Next we show that D_0 and D_1 map $\tilde{\gamma}$ to the same arc. As in the previous Lemma, if p^\pm are the endpoints of $\tilde{\gamma}$, we can extend D_i to p^\pm and $D_0(p^\pm) = D_1(p^\pm)$. Since $\tilde{\gamma} \subset \tilde{\lambda}$ either D_0 and D_1 map $\tilde{\gamma}$ to the same arc or the closure of their images is all of $\mathbb{R} \cup \infty$. Let \tilde{X} denote a connected component of $\tilde{S} \setminus \tilde{\lambda}$ whose boundary contains $\tilde{\gamma}$. If $\overline{D_0(\tilde{\gamma})} \cup \overline{D_1(\tilde{\gamma})} = \mathbb{R} \cup \infty$, then of C_x is mapped to opposite half planes by D_0 and D_1 . This is a contradiction of Lemma 3.10, and so we have $D_0(\tilde{\gamma}) = D_1(\tilde{\gamma})$. So if $x \in \tilde{\lambda}$ and $\tilde{\gamma} \subset \tilde{\lambda}$ contains x , the map

$$\tilde{f}(x) = D_1^{-1}(D_0(x)) \cap \tilde{\gamma}$$

is well defined.

Now we extend \tilde{f} to all of \tilde{S} . For any point $x \in \tilde{S} \setminus \tilde{\lambda}$ let C_x denote the connected component of $\tilde{S} \setminus \tilde{\lambda}$ containing x . By Lemma 3.5 the D_i are homeomorphisms from C_x to Ω_+ or Ω_- . For any $x \in \tilde{S}$ there is a unique point $y \in C_x$ such that $D_0(x) = D_1(y)$. Therefore we define for every $x \in \tilde{S}$, a map $\tilde{f} : \tilde{S} \rightarrow \tilde{S}$ by

$$\tilde{f}(x) = (D_1)^{-1}(D_0(x)) \cap C_x.$$

The continuity of D_i implies that \tilde{f} is continuous on each component of $\tilde{S} \setminus \tilde{\lambda}$. Since D_i is a local homeomorphism, if $\{x_i\} \in C_x$ and $\{y_i\} \in C_y$ converge to $x \in \tilde{\lambda}_0$ then $\{D_0(x_i)\}$ and $\{D_0(y_i)\}$ converge to some point $r \in \mathbb{R} \cup \infty$. Then

$\{D_1^{-1}(D_0(x_i))\} \cap C_x$ and $\{D_1^{-1}(D_0(y_i))\} \cap C_y$ have subsequences which converge to $D_1^{-1}(r) \in \tilde{\lambda}$. Therefore \tilde{f} is continuous on all of \tilde{S} . Then, for any $x \in \tilde{S}$

$$D_1 \circ \tilde{f}(x) = D_1 \circ ((D_1)^{-1}(D_0(x))) \cap C_x = D_0(x).$$

Next we show that \tilde{f} is equivariant with respect to $\pi_i(S)$. For any $g \in \pi_1(S)$ the equivariance of the developing map implies

$$\tilde{f}(gx) = D_1^{-1}(D_0(gx)) \cap C_{gx} = D_1^{-1}(\rho(g)(D_0(x))) \cap C_{gx} = g(D_1^{-1}(D_0(x)) \cap C_x) = g\tilde{f}(x)$$

and therefore \tilde{f} is equivariant with respect to the action of $\pi_1(S)$, and thus \tilde{f} descends to a map $f : S \rightarrow S$.

Finally, since the D_i are equivalent on the endpoints of λ the map \tilde{f} extends to the identity on the boundary of \tilde{S} . By Lemma 3.8 f is isotopic to the identity. \square

We now complete the proof of Theorem 3.7. Let Σ_0 and Σ_1 be projective structures with holonomy ρ . By definition, Σ_i is an equivalence class of pairs (D_i, ρ) where D_i is unique up to precomposition with a lift of a homeomorphism of S that is isotopic to the identity and postcomposition with a Mobius transformation and ρ is unique up to conjugation. Let D_0 and D_1 be developing maps of Σ_0 and Σ_1 such that λ_0 is homotopic to λ_1 . There is then a map $h_1 : S \rightarrow S$ isotopic to the identity for which $h_1(\lambda_0) = \lambda_1$. Let \tilde{h}_1 be a lift of h_1 to \tilde{S} . Then $D'_0 = D_0 \circ \tilde{h}_1$ is equivalent to D_0 .

By Lemma 3.11 there is a map $h_2 : S \rightarrow S$ isotopic to the identity which has a lift \tilde{h}_2 such that $D'_0 = D_1 \circ \tilde{h}_2$ is equivalent to D_1 . Then D_1 is equivalent to D_0 and therefore $\Sigma_0 = \Sigma_1$.

Notation 3.12 Let $\Sigma = (D, \rho)$ be a projective structure on S . By Theorem 3.7 we may denote Σ by $\Sigma(\rho, \lambda)$ (or alternatively (ρ, λ)).

Definition 3.13 Let γ be a simple closed curve in S . Denote by $[\gamma]$ the homotopy class of γ . If δ is an admissible curve in $[\gamma]$ then we say δ is an admissible representative of $[\gamma]$.

Proposition 3.14 Let $\Sigma(\rho, \lambda)$ be a projective structure with a real representation ρ . If γ is an admissible representative of $[\gamma]$ disjoint from the real curves λ of $\Sigma(\rho, \lambda)$ then

$$Gr_\gamma(\Sigma(\rho, \lambda)) = \Sigma(\rho, \lambda \cup 2\gamma)$$

where 2γ denotes two simple closed curves isotopic to γ .

Proof : By definition, grafting $\Sigma(\rho, \lambda)$ along γ produces a new projective structure $(D_1, \rho) = \Sigma(\rho, \lambda_\gamma)$. Since grafting affects the structure locally near γ , the fact that γ is disjoint from λ implies that $D_1(\tilde{\lambda}) = D(\tilde{\lambda}) = \mathbb{R}$ and so $\lambda \subset \lambda_\gamma$.

Let A denote the annulus grafted into $\Sigma(\rho, \lambda)$ along γ , and let L be a component of \tilde{A} . There is some $g \in \pi_1(S)$ which fixes L and thus $\rho(g)$ fixes the annulus $D_1(L)$ in $\hat{\mathbb{C}}$. The fixed points $\{p_1, p_2\}$ of $\rho(g)$ form the boundary of $D_1(L)$ and split $\mathbb{R} \cup \infty$ into two components that are both contained in $D_1(L)$.

Then, in L there are two arcs $\tilde{\gamma}_1, \tilde{\gamma}_2$ of $D_1^{-1}(\mathbb{R})$ and the boundaries of these arcs are the fixed points of g . Therefore $\gamma_i = \tilde{\gamma}_i/g$ are simple closed curves in A isotopic to γ and

$$\lambda_\gamma = \lambda \cup \gamma_1 \cup \gamma_2.$$

□

Assume α and β are admissible representatives for $[\alpha]$ and $[\beta]$ in a projective structure $\Sigma = (\rho, \lambda)$, and

$$\alpha \cap \beta = \alpha \cap \lambda = \beta \cap \lambda = \phi.$$

By Proposition 3.14 $Gr_\alpha(\rho, \lambda) = (\rho, \lambda \cup 2\alpha)$ and $Gr_\beta(\rho, \lambda) = (\rho, \lambda \cup 2\beta)$. By Lemma 3.6 α has an admissible representative in $(\rho, \lambda \cup 2\beta)$ and β has an admissible representative in $(\rho, \lambda \cup 2\alpha)$. Proposition 3.14 again implies $Gr_\alpha(\rho, \lambda \cup 2\beta) = (\rho, \lambda \cup 2\beta \cup 2\alpha)$ and $Gr_\beta(\rho, \lambda \cup 2\alpha) = (\rho, \lambda \cup 2\alpha \cup 2\beta)$. Then Theorem 3.7 implies

$$Gr_\alpha(Gr_\beta(\rho, \lambda)) = Gr_\beta(Gr_\alpha(\rho, \lambda)).$$

Let $\sigma = \{\alpha_i\}$ denote a (possibly empty) collection of disjoint admissible curves in (ρ, λ) which are each disjoint from λ . By the above argument we may denote

$$Gr_\sigma(\Sigma) = Gr_{\alpha_1} \circ Gr_{\alpha_2} \circ \dots \circ Gr_{\alpha_n}(\Sigma)$$

4 Fuchsian Holonomy

In this section we consider the case that ρ is a faithful representation onto a Fuchsian group. We formulate a proof of a theorem of Goldman, which uses the above topological ideas instead of the algebraic machinery used in [7].

Definition 4.1 *A projective structure Σ is standard if its developing map is a covering map onto its image. Equivalently, the projective structure is the quotient of the domain of discontinuity by the holonomy representation*

$$\Sigma = \Omega / \rho.$$

Theorem 4.2 (Goldman) *If $\Sigma = (\rho, \lambda)$ is a complex projective structure with ρ a Fuchsian representation, then there exists a collection of admissible disjoint simple closed curves σ such that*

$$\Sigma = Gr_\sigma(\Sigma_0)$$

where Σ_0 is a standard (hyperbolic) complex projective structure.

Proof : Any Fuchsian projective structure $\Sigma = (\rho, \lambda)$ has the property that each simple closed curve in λ contains an even number of curves in its isotopy class. This follows from Lemma 3.9 in the following way.

Let $\Sigma_0 = (\rho, \lambda)$ be a standard Fuchsian projective structure. Let $U \subset S$ be any non-annular region disjoint from λ . Since Σ_0 is standard and ρ is Fuchsian,

U is disjoint from $\lambda_0 = \phi$. Now Lemma 3.9 implies that $D(U)$ and $D_0(U)$ are both positive (contained in Ω_+) or negative (contained in Ω_-).

This implies that every non-annular region in S that is disjoint from λ is, say, positive. Since adjacent regions of $S \setminus \lambda$ are mapped to opposite half planes the number of disjoint, simple closed isotopic curves in λ must be even. Since S is compact the number of such curves in λ is finite.

Enumerate the real curves of Σ as $\lambda = 2\gamma_1 \cup 2\gamma_2 \cup \dots \cup 2\gamma_n$. Let $\sigma = \{\delta_1 \dots \delta_n\}$ denote a collection of n disjoint curves in S , where δ_i is isotopic to γ_i . Since Σ_0 is standard $\delta_i \cap \lambda_0 = \phi$ so Lemma 3.6 implies that δ_i is admissible. By Proposition 3.14

$$Gr_\sigma(\Sigma_0) = (\rho, 2\delta_1 \cup 2\delta_2 \cup \dots \cup 2\delta_n)$$

and Theorem 3.7 then implies $Gr_\sigma(\Sigma_0) = \Sigma$. \square

5 Old Chapter 4

6 Real Schottky Holonomy

In this section we consider real complex projective structures whose holonomy representation is onto a Schottky subgroup of $PSL_2(\widehat{\mathbb{C}})$. Since these representations are real, we will use ideas developed in the previous section. We define a Schottky group as in [13].

Definition 6.1 *Let $B_1, \dots, B_k, B'_1, \dots, B'_k$ be disjoint closed geometric disks in $\widehat{\mathbb{C}}$, and let $int(B_i)$ denote the interior of B_i and $ext(B_i)$ denote the exterior of B_i . Let g_i be a Mobius transformation such that $g_i: int(B_i) \rightarrow ext(B'_i)$. The group $\Gamma = \langle g_1, g_2, \dots, g_k \rangle$ is called a classical Schottky group.*

From here forward we fix a representation $\rho: \pi_1(S) \rightarrow \Gamma$ where $\Gamma < PSL_2(\mathbb{R})$ is a real Schottky group. We also assume that all curves intersect transversely. Theorem 3.7 implies that our study of grafted real projective structures is reduced to a study of the behavior of their real curves near the grafting curve. With this in mind we begin with a careful examination of the universal cover of the Hopf torus and its intersection with the real line.

7 Basic Facts for Curves in a Torus

Let (α, β) be an ordered pair of essential oriented transverse curves in a torus T which intersect exactly once. Then (α, β) determines an orientation for T and we choose this orientation for T . Given any orientation of T we define the index of an intersection point of two curves α and β to be $+1$ if the orientation on $T_x(T)$ induced by (α, β) agrees with the given orientation of T , and -1 otherwise. The algebraic intersection number $\hat{i}(\alpha, \beta)$ is defined to be the sum of the indices of all the intersections of α and β . Since $H_1(T) \simeq \mathbb{Z} \oplus \mathbb{Z}$, given a basis for H_1 the homology class of any simple closed multicurve is given by

an ordered pair of integers (p, q) . An orientation on a simple closed curve in T is a choice of sign of the associated integer pair thus (p, q) has the opposite orientation of $(-p, -q)$.

8 The Hopf Torus

Next we consider a particular class of admissible curves in S whose associated graftings can be described by the image of these curves in the Hopf torus. A lift of such a curve spirals around its fixed points as suggested in Figure 2. This reduces the study of projective structures grafted along such curves to a study of curves in a torus.

Definition 8.1 *Choose an admissible representative γ and conjugate ρ so that $Fix(\rho(\gamma)) = \{0, \infty\}$. Let $\tilde{\gamma}$ be a component of the preimage of γ in the universal cover of γ , and set $\hat{\gamma} = D(\tilde{\gamma})$. Then γ is spiraling if consecutive points of $\hat{\gamma} \cap \mathbb{R}$ alternate between \mathbb{R}^- and \mathbb{R}^+ .*

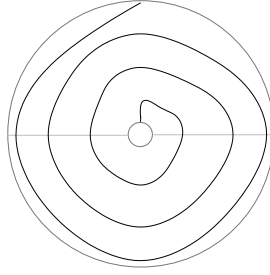


Figure 2: Left-spiraling

Let $\gamma_T = \hat{\gamma}/\rho(\gamma)$ denote the projection of γ to the Hopf torus. We claim that γ_T has minimal intersection with $\mathbb{R}/\rho(\gamma)$. Assume γ_T does not have minimal intersection with $\mathbb{R}/\rho(\gamma)$. Then there is an innermost disk in T_H bounded by $\mathbb{R}/\rho(\gamma)$. Since \mathbb{R}^- and \mathbb{R}^+ are disjoint, any such disk lifts to a disk in $\hat{\mathbb{C}}$ whose boundary is composed of two arcs, one from $\hat{\gamma}$ and one from either \mathbb{R}^- or \mathbb{R}^+ . Then consecutive points of $\hat{\gamma} \cap \mathbb{R}$ lie on a single component of $\mathbb{R} \setminus \{0, \infty\}$ and γ cannot be spiraling.

Recall, the Hopf torus T_H is defined as the quotient $(\hat{\mathbb{C}} \setminus Fix(\rho(\gamma)))/\rho(\gamma)$. So any curve in S with holonomy $\rho(\gamma)$ projects to an essential closed curve in T_H . Let α be a geometric circle in $\hat{\mathbb{C}}$ which separates $Fix(\rho(\gamma))$. Since $Fix(\rho(\gamma)) = \{0, \infty\}$ it follows that $|\alpha \cap \mathbb{R}^-| = 1$ and thus $|\alpha_T \cap \lambda_-| = 1$ where $\alpha_T = \alpha/\rho(\gamma)$ and $\lambda_- = \mathbb{R}^-/\rho(\gamma)$.

We orient α_T and λ_- so that the orientation of T produced by the oriented ordered pair (λ_-, α_T) agrees with the standard orientation of T . We choose as a basis for $H_1(T_H)$ the ordered pair of oriented simple closed curves (λ_-, α_T) . Now $\lambda_- = (1, 0)$ and $\alpha_T = (0, 1)$.

Since α separates the endpoints of $\hat{\gamma}$ we have $\hat{i}(\hat{\gamma}, \alpha) = 1$. So in the Hopf torus $\hat{i}(\gamma_T, \alpha_T) = 1$. Since γ_T is closed

$$\gamma_T = (1, k)$$

for $k \in \mathbb{Z}$ and we say γ_T is left-spiraling if $k > 0$ and is right-spiraling if $k < 0$.

9 Flat and Sharp Operations

Next we define a surgery operation on curves in a surface. We will later express the real curves obtained by grafting along spiraling curves as the result of this surgery on the real curves of the initial structure together with curves isotopic to the grafting curve. The surgery is depicted in Figure 3.

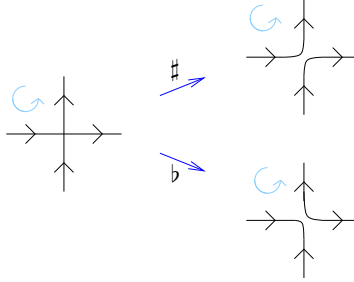


Figure 3: Flat and sharp operations

Let (λ, γ) be an ordered pair of simple closed multicurves in oriented surface S . For each $x \in \lambda \cap \gamma$ let $N(x)$ denote a neighborhood of x in S so that $N(x) \cap (\lambda \cap \gamma) = x$. Choose a local orientation of λ and γ near each $x \in \lambda \cap \gamma$ such that the orientation induced by (λ, γ) agrees with the orientation of S .

The multicurve $[\lambda, \gamma]_{\#}$ is obtained by resolving each $x \in \lambda \cap \gamma$ by replacing x with arcs in $N(x)$ joining λ to γ so that the local orientations of the arcs being joined agree. The \flat operation joins arcs whose local orientations disagree. If λ and γ are given the opposite orientation the same topological curves are obtained, but with the opposite orientation.

9.1 Operations on the Torus

It will often be the case that these operations on the surface S may be reduced to the same operations on the Hopf torus. Here we derive relations between $\#/\flat$ operations in the case of the torus. Recall that we assume all intersections are transverse and minimal.

Lemma 9.1 *Let (α, β) be an ordered pair of essential simple closed curves in an oriented surface S . Then*

$$[\alpha, \beta]_{\#} = [\beta, \alpha]_{\flat}.$$

Proof :

The local orientations required to compute $[\beta, \alpha]_{\flat}$ may be obtained from those required to compute $[\alpha, \beta]_{\#}$ by reversing the local orientations of exactly one of the curves near each x_i . Resolving intersections so that the orientations of the curves being joined disagree is the same as first reversing the orientations of one of the curves and then resolving intersections so that orientations of the curves being joined agree. Therefore the first and second computation produce the same set of curves. \square

Assume α and β are oriented essential simple closed curves in T in minimal position. The $\#$ operation depends only on the orientation of the surface, not on the orientations of the curves themselves, but we will use the orientation α to define an orientation of $[\alpha, \beta]_{\#}$ and $[\alpha, \beta]_{\flat}$.

Since α and β are curves in a torus, each index of an intersection point of α with β is the same for each point. Choose an orientation of β so that each index is +1. Then by definition the $\#$ operation joins α and β so that the orientations agree, and we choose this orientation for $[\alpha, \beta]_{\#}$. Then $[\alpha, \beta]_{\flat}$ inherits an orientation in a similar manner if we choose an orientation of β so that each index is -1.

Lemma 9.2 *Assume that α, β and γ are oriented simple closed curves in an oriented torus T . Then $\hat{i}(\alpha, \beta) > 0$ implies*

$$\begin{aligned}\hat{i}([\alpha, \beta]_{\#}, \gamma) &= \hat{i}(\alpha, \gamma) + \hat{i}(\beta, \gamma) \\ \hat{i}([\alpha, \beta]_{\flat}, \gamma) &= \hat{i}(\alpha, \gamma) - \hat{i}(\beta, \gamma)\end{aligned}$$

and $\hat{i}(\alpha, \beta) < 0$ implies

$$\begin{aligned}\hat{i}([\alpha, \beta]_{\flat}, \gamma) &= \hat{i}(\alpha, \gamma) + \hat{i}(\beta, \gamma) \\ \hat{i}([\alpha, \beta]_{\sharp}, \gamma) &= \hat{i}(\alpha, \gamma) - \hat{i}(\beta, \gamma)\end{aligned}$$

Proof : Transversality implies that the intersection of α and β is disjoint from γ . We may then assume that each point of $\gamma \cap [\alpha, \beta]_{\sharp}$ corresponds exactly with either a point of $\gamma \cap \alpha$ or $\gamma \cap \beta$. The assumption that $\hat{i}(\alpha, \beta) > 0$ implies that at each point of intersection of α and β the orientations of the curves agree with the orientation of T . Therefore the orientation of $[\alpha, \beta]_{\sharp}$ is the same as α where they coincide, and similarly for β . Therefore

$$\hat{i}([\alpha, \beta]_{\sharp}, \gamma) = \hat{i}(\alpha, \gamma) + \hat{i}(\beta, \gamma).$$

Since $[\alpha, \beta]_{\sharp} = [\alpha, -\beta]_{\flat}$ it follows from $\hat{i}(-\beta, \gamma) = -\hat{i}(\beta, \gamma)$ that

$$\hat{i}([\alpha, \beta]_{\flat}, \gamma) = \hat{i}(\alpha, \gamma) - \hat{i}(\beta, \gamma).$$

If $\hat{i}(\alpha, \beta) < 0$ then $\hat{i}(\alpha, -\beta) > 0$ so the argument above implies

$$\hat{i}([\alpha, \beta]_{\flat}, \gamma) = \hat{i}(\alpha, \gamma) - \hat{i}(-\beta, \gamma) = \hat{i}(\alpha, \gamma) + \hat{i}(\beta, \gamma)$$

and

$$\hat{i}([\alpha, \beta]_{\sharp}, \gamma) = \hat{i}(\alpha, \gamma) + \hat{i}(-\beta, \gamma) = \hat{i}(\alpha, \gamma) - \hat{i}(\beta, \gamma).$$

□

Above the ordered pair of integers (p, q) denotes a homology class of an oriented simple closed multicurve in a torus. We will also use this same notation to denote a representative curve from the homology class.

Lemma 9.3 *Let (p, q) and (r, s) denote two oriented essential simple closed curves in an oriented torus T . Then $\hat{i}(\alpha, \beta) \geq 0$ implies*

$$[(p, q), (r, s)]_{\sharp} = (p + r, q + s)$$

and

$$[(p, q), (r, s)]_{\flat} = (p - r, q - s)$$

and $\hat{i}(\alpha, \beta) < 0$ implies

$$[(p, q), (r, s)]_{\flat} = (p + r, q + s)$$

and

$$[(p, q), (r, s)]_{\sharp} = (p - r, q - s).$$

Proof : If $\hat{i}((p, q), (r, s)) = 0$ then the curves are disjoint and are thus homotopic, so the Lemma holds in this case. Assume $\hat{i}((p, q), (r, s)) > 0$.

By Lemma 9.2

$$\hat{i}([(p, q), (r, s)]_{\sharp}, (1, 0)) = \hat{i}((p, q), (1, 0)) + \hat{i}((r, s), (1, 0)) = -q - s$$

and

$$\hat{i}([(p, q), (r, s)]_{\#}, (0, 1)) = \hat{i}((p, q), (0, 1)) + \hat{i}((r, s), (0, 1)) = p + r.$$

Then $[(p, q), (r, s)]_{\#} = (p + r, q + s)$. Since $[\alpha, \beta]_{\flat} = [\alpha, -\beta]_{\#}$ it follows that

$$[(p, q), (r, s)]_{\flat} = ((p, q), -(r, s))_{\#} = [(p, q), (-r, -s)]_{\#} = (p - r, q - s).$$

A similar proof works for the case that $\hat{i}((p, q), (r, s)) < 0$. □

10 Proof of the Main Theorems

Lemma 10.1 *Let α , γ and λ be oriented simple closed curves in an oriented surface S . Assume α and λ are homotopic relative their boundary in the complement of an annulus A which is homeomorphic to a regular neighborhood of γ . Let $\psi : A \rightarrow T$ be a map onto a torus which identifies the boundary components of A so that the image under ψ of each component of $\alpha \cap A$, $\lambda \cap A$ and $\gamma \cap A$ is a simple closed curve in T . Then*

$$\alpha_T \simeq [\lambda_T, 2\gamma_T]_{\#} \Leftrightarrow \alpha \simeq [\lambda, 2\gamma]_{\#}$$

and

$$\alpha_T \simeq [\lambda_T, 2\gamma_T]_{\flat} \Leftrightarrow \alpha \simeq [\lambda, 2\gamma]_{\flat}$$

where $\alpha_T = \psi(\alpha \cap A)$, $\lambda_T = \psi(\lambda \cap A)$ and $\gamma_T = \psi(\gamma \cap A)$.

Proof : By symmetry of the $\flat/\#$ operations we may work with just one, say \flat . Let 2γ denote two disjoint curves isotopic to γ , and assume $2\gamma \subset A$ and therefore all intersections of $2\gamma \cap \lambda$ are contained in the interior of A .

The \flat operation affects 2γ and λ only near their intersection and ψ is a quotient map on A which defines T so it follows that

$$([\lambda, 2\gamma]_{\flat} \cap A)/\psi = [\lambda_T, 2\gamma_T]_{\flat}.$$

First we assume $[\lambda_T, 2\gamma_T]_{\flat} \simeq \alpha_T$. Then by definition of α_T , $[\lambda_T, 2\gamma_T]_{\flat} \simeq (\alpha \cap A)/\psi$, and the statement above implies

$$([\lambda, 2\gamma]_{\flat} \cap A) \simeq \alpha \cap A$$

where homotopy here is relative to the boundary. And since $\alpha \simeq \lambda$ outside of A this now implies

$$[\lambda, 2\gamma]_{\flat} \simeq \alpha.$$

Now assume $\alpha \simeq [\lambda, 2\gamma]_{\flat}$. The assumption that $\alpha \setminus A \simeq \lambda \setminus A$ now implies that $\alpha \cap A \simeq [\lambda, 2\gamma]_{\flat} \cap A$ where here again is homotopy relative to the boundary. Therefore $\alpha_T \simeq [\lambda_T, 2\gamma_T]_{\flat}$

□

10.1 The Hopf Torus

In the next Lemma it will be helpful to compare a curve λ on the surface S with its developed image before and after grafting. Topologically, the comparison is very simple and is obtained using the following definition.

Definition 10.2 *Let δ be a closed curve in $\Sigma(\rho, \lambda)$. The topological image δ' of δ under grafting along a curve γ , is defined as*

$$\delta_{\gamma} = \nu^{-1}(\delta)$$

where ν is the collapsing map defined in Section 2.4.

Set $\lambda' = \nu^{-1}(\lambda)$. By the construction of h in 2.4

$$\lambda_T = D(\tilde{\lambda}' \cap \tilde{A})/\rho(g) = (0, 2k)$$

with $k > 0$ in the basis (λ_-, α_T) for T .

Assume γ is an admissible representative in $\Sigma(\rho, \lambda) = (D, \rho)$. Let A denote the grafted annulus in $Gr_\gamma(\Sigma(\rho, \lambda)) = (D', \rho)$ and let λ' be the preimage of λ under grafting by γ . Let $\tilde{\lambda}'$ and \tilde{A} denote components of the preimage of λ' and A in \tilde{S} which intersect. Let $g \in \pi_1(S)$ be the element that fixes \tilde{A} .

The real multicurve λ is separating therefore $|\lambda \cap \gamma| = 2k$ for some nonnegative integer k . By Definition 10.2

$$\nu(\lambda' \cap A) = x_i \in \lambda$$

for $i \in [1, 2k]$. Since our choice of collapsing map (Section 2.4) ν collapses arcs in A which develop to simple closed curves, $D(\tilde{\lambda}' \cap \tilde{A})$ consists of $2k$ simple closed curves in $\hat{\mathbb{C}} - Fix(\rho(\gamma))$. These arcs are contained in a fundamental domain of $\hat{\mathbb{C}} - Fix(\rho(\gamma))$ for the action of $\rho(g)$.

Theorem 10.3 *Let $\Sigma = \Sigma(\rho, \lambda)$ be a real Schottky projective structure on S . Let γ be an admissible right-spiraling representative in S . Then*

$$Gr_\gamma(\Sigma(\rho, \lambda)) = \Sigma(\rho, \lambda_\gamma)$$

where

$$\lambda_\gamma \simeq [\lambda, 2\gamma]_b$$

where 2γ denotes two copies of γ .

Proof :

By construction of the collapsing map, $\lambda_T = \delta_\gamma(\lambda)/\rho(g)$ is a $(0, 2k)$ multicurve with $k > 0$ in the Hopf torus T with basis (λ_-, α_T) . By definition of right-spiraling $\gamma_T = (1, -k)$ in T and we let $2\gamma_T = (2, -2k)$ denote two copies of γ_T in T . Then since $\hat{i}((0, 2k), (2, -2k)) = -4k$, Lemma 9.3 implies

$$[\lambda_T, 2\gamma_T]_b = (2, 0)$$

in T . Since λ_γ develops to \mathbb{R} under the new grafted developing map,

$$h(\lambda_\gamma \cap A) = (2, 0) = 2\alpha_T$$

in T . And since grafting affects the real curves only in a neighborhood of the grafting curve λ_γ and $\nu^{-1}(\lambda)$ are homotopic in the complement of A so Lemma 10.1 implies

$$\lambda_\gamma = [\nu^{-1}(\lambda), 2\gamma]_b \simeq [\lambda, 2\gamma]_b.$$

Similarly, if γ is left-spiraling then $2\gamma_T = (2, 2k)$, and thus

$$\lambda_\gamma = [\lambda, 2\gamma]_b.$$

□

In Figure 4 a fundamental domain for a component of the universal cover of the grafted annulus is depicted along with its intersection with lifts of the newly obtained real curves resulting from grafting. The horizontal curves entering the strip are the original real curves while the curves inside are new real curves produced by grafting.



Figure 4: A lift of the grafting annulus and preimage of the real curves

11 Grafting Complex of Real Complex Projective Structures

Definition 11.1 *We say two projective structures differ by an elementary move if their real laminations differ by a Dehn twist along a meridian curve which intersects one of the laminations exactly twice.*

Let $\mathcal{G}(S)$ be a complex of Fuchsian Schottky projective structures where each vertex is a structure, and an edge between two vertices exists if the two corresponding structures differ by an elementary move or by grafting. In the remainder of this section unless explicitly stated otherwise let β, γ, λ be essential simple closed multicurves in S such that $|\beta \cap \gamma| = 1$, $|\beta \cap \lambda| = 2$. Let k be an integer and T_β^k denote k Dehn twists about β .

In [5] Chapter 3 page 64, the authors observe that the effect on curves of a (right) Dehn twist is realized via a surgery operation that is equivalent to our \flat operation. If a and b are simple closed curves, to realize $T_a(b)$ the set of curves $i(a, b)a \cup b$ is surgered so that if one follows an arc of b towards the intersection, the surgered arc turns right at the intersection. This is precisely our \flat operation so the following relationship holds between Dehn twists and \flat operation.

Lemma 11.2 *If α and β are essential simple closed curves with minimal intersection in a surface S then*

$$T_\alpha(\beta) = [\beta, i(\alpha, \beta)\alpha]_\flat$$

and

$$T_\alpha^{-1}(\beta) = [\beta, i(\alpha, \beta)\alpha]_\sharp$$

where $i(\alpha, \beta)$ is the geometric intersection number.

If γ and γ' are oriented admissible curves and $\rho(\gamma) = \rho(\gamma')$, we say that they are oriented *consistently* if their respective developed images have components γ_D and γ'_D such that with the induced orientation they have the same initial endpoint.

Lemma 11.3 Assume γ , γ' and λ are simple closed curves in a standard projective structure $\Sigma(\rho, \lambda)$ such that $\gamma \cap \lambda = \phi$, $\rho(\gamma) = \rho(\gamma')$, γ and γ' are oriented consistently and

$$|\hat{i}(\gamma, \gamma')| = i(\gamma, \gamma') = \frac{1}{2}i(\gamma', \lambda).$$

If for every arc $a' \in \gamma' - \gamma$ there is an arc $a \in \gamma - \gamma'$ such that the curve $a' \cup a$ has trivial holonomy, then γ' is spiraling and admissible in $\Sigma(\rho, \lambda)$.

Proof :

By Lemma 3.6, γ is admissible. The standard developing map is a covering map onto its image so γ' is admissible since it is simple. Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be components of the universal cover of γ and γ' which intersect.

Let a be an arc of $\gamma' - \gamma$. Since λ is the real curve of $\Sigma(\rho, \lambda)$ it is a separating curve so the arc a must cross λ an even number of times. If $a \cap \lambda$ consisted of more than two points some other arc of $\gamma' - \gamma$ would have to be disjoint from λ since $i(\gamma, \gamma') = \frac{1}{2}i(\gamma', \lambda)$.

Let $\tilde{a} \subset \hat{\mathbb{C}}$ be the developed image of a component of $\pi^{-1}(a)$ where $\pi : \tilde{S} \rightarrow S$ is the universal covering. Let $D(\tilde{\gamma})$ be the developed image of a component of $\pi^{-1}(\gamma)$ such that the arc \tilde{a} has both endpoints on $D(\tilde{\gamma})$. Such a $D(\tilde{\gamma})$ exists by the assumption that $a' \cup a$ has trivial holonomy.

Assume $D(\tilde{\gamma})$ has endpoints $\{0, \infty\}$ and is oriented from 0 to ∞ . Then $D(\tilde{\gamma})$ splits the upper half plane into two regions, called region 0 (which is bounded by R^+ and $D(\tilde{\gamma})$) and region 1 (which is bounded by R^{-1} and $D(\tilde{\gamma})$). The lower half plane we call region 2. The assumption that $|\hat{i}(\gamma, \gamma')| = i(\gamma, \gamma')$ implies that all indices of $D(\tilde{\gamma}) \cap D(\tilde{\gamma}')$ are the same. Assume $\hat{i}(\gamma, \gamma') > 0$. Then the oriented arc \tilde{a} starts in region 1 and ends in region 0 and \tilde{a} must pass through region 2 since it cannot intersect $D(\tilde{\gamma})$. This implies that \tilde{a} intersects \mathbb{R} and therefore a must intersect λ . By the comment above, since every arc of $\gamma' - \gamma$ intersects λ it must do so exactly twice. The first point of $\tilde{a} \cap \mathbb{R}$ is negative and the last point is positive since \tilde{a} starts in region 1 and ends in region 0. Thus γ' is right-spiraling. If $\hat{i}(\gamma, \gamma') < 0$ the argument above implies that γ' is left-spiraling. \square

Lemma 11.4 Assume $\gamma \cap \lambda = \phi$ and β is a collection of disjoint meridians in S such that $|\gamma \cap \beta_i| = 1$ and $|\lambda \cap \beta_i| = 2$. Then $T_\beta^n(\gamma)$ has a left (or right) admissible spiraling representative in the standard projective structure $\Sigma(\rho, \lambda)$ for $n \in \mathbb{Z}^+$ (or $n \in \mathbb{Z}^-$).

Proof :

Since γ is essential, Lemma 3.6 implies γ is admissible in $\Sigma(\rho, \lambda)$. Let S have the orientation induced by the standard orientation of \mathbb{C} . Since each β_i is a meridian in S , $\rho(\beta_i) = 1$ and therefore $\rho(T_\beta^n(\gamma)) = \rho(\gamma)$. The fact that $|\gamma \cap \beta_i| = 1$ implies that $T_{\beta_i}^n(\gamma) = [\gamma, n\beta_i]_b$. After noting that $\hat{i}(\gamma, T_\beta^n(\gamma)) = -n$ we have

$$|\hat{i}(T_\beta^n(\gamma), \gamma)| = i(T_\beta^n(\gamma), \gamma) = n|\beta|$$

where $|\beta|$ is the number of components of β .

Since $|\lambda \cap \beta_i| = 2$ each twist of γ about β adds 2 intersection points to $i(T_{\beta_i}^n(\gamma), \lambda)$, so

$$i(T_{\beta_i}^n(\gamma), \lambda) = 2n|\beta|.$$

In [11] Ito observed that $[\gamma, T_{\beta}^n(\gamma)]_{\#} = [\gamma, [\gamma, n\beta]_{\flat}]_{\#} = n\beta$. There are exactly $n|\beta|$ arcs of both $T_{\beta}^n(\gamma) - \gamma$ and $\gamma - T_{\beta}^n(\gamma)$. Therefore each component of $n\beta$ is composed of exactly 1 arc of $T_{\beta}^n(\gamma) - \gamma$ and one arc of $\gamma - T_{\beta}^n(\gamma)$. This implies that for every arc of $T_{\beta}^n(\gamma) - \gamma$ there is an arc of $\gamma - T_{\beta}^n(\gamma)$ whose union has trivial holonomy, see Figure 5. Since a Dehn twist is an orientation preserving homeomorphism for any orientation of γ , then with the induced orientation on $T_{\beta}^n(\gamma)$, γ and $T_{\beta}^n(\gamma)$ are oriented consistently.

Now we can apply Lemma 11.3 to see that $T_{\beta}^n(\gamma)$ is spiraling. Since $\hat{i}(\gamma, T_{\beta}^n(\gamma)) = -n$, Lemma 11.3 implies that if $n > 0$, $T_{\beta_i}^n(\gamma)$ is left-spiraling, and if $n < 0$, $T_{\beta_i}^n(\gamma)$ is right-spiraling. □

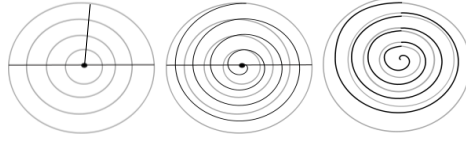


Figure 5: The effect of two Dehn twists of γ about a meridian

Next we use Theorem 10.3 with Lemma 11.2 to show that in some cases the real lamination produced by grafting is a Dehn twist of the original lamination along the grafting curve.

Corollary 11.5 *Suppose γ is an admissible right-spiraling representative for which*

$|\lambda \cap \gamma| = 2$. Then

$$Gr_{\gamma}(\Sigma(\rho, \lambda)) = \Sigma(\rho, T_{\gamma}(\lambda)).$$

If γ is an admissible left-spiraling representative for which $|\lambda \cap \gamma| = 2$. Then

$$Gr_{\gamma}(\Sigma(\rho, \lambda)) = \Sigma(\rho, T_{\gamma}^{-1}(\lambda)).$$

Proof :

Theorem 10.3 implies the new real curves are given by $\lambda_{\gamma} \simeq (\lambda, 2\gamma)_{\flat}$ in the case that γ is right-spiraling and $\lambda_{\gamma} \simeq (\lambda, 2\gamma)_{\#}$ if γ is left-spiraling. Since $|\lambda \cap \gamma| = 2$, Lemma 11.2 implies $T_{\gamma}(\lambda) = (\lambda, 2\gamma)_{\flat}$ if γ is right-spiraling and $T_{\gamma}(\lambda) = (\lambda, 2\gamma)_{\#}$ if γ is left-spiraling. □

Our next goal is to show that for every two structures that differ by a power of an elementary move (via T_{β}^k), there is an infinite number of structures realized as a graft of each. How this is done is suggested in Figure 6.

The next few computations will make use of the following formula for the geometric intersection number $i(a, b)$ for curves a and b on a torus [5].

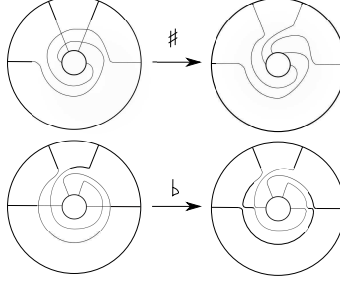


Figure 6: The annulus A where $[\lambda, T^k(2\gamma)]_{\#} \simeq [T^k(\lambda), 2\gamma]_{\flat}$ for $k = -1$

$$i(T_{(p,q)}^k(r, s), (r, s)) = |k| \cdot |ps - qr|^2.$$

Lemma 11.6 *Let β, γ, λ be simple closed multicurves in S . Assume $|\beta \cap \lambda| = 2$, $|\beta \cap \gamma| = 1$ and $|\gamma \cap \lambda| = 0$. Set $T^k = T_{\beta_T}^k$ for $k \in \mathbb{Z}$.*

Then

$$[\lambda, T^k(2\gamma)]_{\#} \simeq [T^k(\lambda), 2\gamma]_{\flat}$$

and

$$T^k[\lambda, 2\gamma]_{\#} \simeq [\lambda, T^{2k}(2\gamma)]_{\#} \simeq [T^{2k}(\lambda), 2\gamma]_{\flat} \simeq [T^k(\lambda), T^k(2\gamma)]_{\#}.$$

Proof :

Since $|\gamma \cap \lambda| = 0$ there is an element $b \in [\beta]$ and a regular neighborhood A of b containing all intersection points $\lambda \cap T^k(\gamma)$. After replacing β with b we may assume that $(\lambda \cap T^{2k}(\gamma))$, $(\lambda \cap T^k(\gamma))$ and $(T^k(\lambda) \cap \gamma)$ are all contained in A . Then for $k \in \mathbb{Z}$ we have

$$[\lambda, T^k(2\gamma)]_{\#}|_{(S \setminus A)} \simeq [T^k(\lambda), 2\gamma]_{\flat}|_{(S \setminus A)} \simeq (\lambda \cup 2\gamma)|_{(S \setminus A)}$$

and

$$[T^{2k}(\lambda), 2\gamma]_{\flat}|_{(S \setminus A)} \simeq T^k[\lambda, 2\gamma]_{\#}|_{(S \setminus A)}.$$

For a curve $\alpha \in S$ let $\alpha_T = f(\alpha \cap A)$ where $f : A \rightarrow T$ is a map of A which identifies ∂A so that $f(\lambda \cap A) = (2, 0)$, $f(\beta \cap A) = (0, 1)$ and $f(\gamma \cap A) = (1, 0)$. Then $T_{\beta_T}^k(\gamma_T) = (1, k)$ and $T_{\beta_T}^k(\lambda_T) = (2, 2k)$.

Now

$$[\lambda_T, T_{\beta_T}^k(2\gamma_T)]_{\#} = [(2, 0), (2, 2k)]_{\#} = (4, 2k).$$

And since

$$[T_{\beta_T}^k(2\gamma_T), \lambda_T]_{\#} = [(2, 2k), (2, 0)]_{\#} = (4, 2k)$$

Lemma 9.1 implies that

$$[\lambda_T, T_{\beta_T}^k(2\gamma_T)]_{\#} = [T_{\beta_T}^k(2\gamma_T), \lambda_T]_{\flat}.$$

Also

$$T^k[\lambda_T, 2\gamma_T]_{\#} = T^k[(2, 0), (2, 0)]_{\#} = T^k(4, 0) = (4, 4k)$$

and

$$[\lambda_T, T^{2k}(2\gamma_T)]_{\#} = [(2, 0), (2, 4k)]_{\#} = (4, 4k)$$

and

$$[T^{2k}(\lambda_T), 2\gamma_T]_{\flat} = [(2, 0), (2, 4k)]_{\#} = (4, 4k)$$

and

$$[T^k(\lambda_T), T^k(2\gamma_T)]_{\#} = [(2, 2k), (2, 2k)]_{\#} = (4, 4k)$$

and so by Lemma 10.1

$$[\lambda, T^{2k}(2\gamma)]_{\#} \simeq [T^{2k}(\lambda), 2\gamma]_{\flat} \simeq T^k[\lambda, 2\gamma]_{\#} \simeq [T^k(\lambda), T^k(2\gamma)]_{\#}$$

□

The next theorem provides a sufficient condition on an admissible curve δ for the real lamination obtained by grafting Σ along δ to be the multicurve $T_i^k(\lambda) \cup 2\delta$. Furthermore, it asserts that for any two structures $\Sigma(\rho, \lambda)$ and $\Sigma(\rho, T_i^k(\lambda))$ differing by an iterated elementary move there are an infinite number of ways (parameterized by $m \in \mathbb{Z}$) to connect them by grafting. Equivalently, there are an infinite number of structures realized as a graft of both $\Sigma(\rho, \lambda)$ and $\Sigma(\rho, T_i^k(\lambda))$.

Theorem 11.7 *Let k, l, m be integers such that $k + l = m$. Let β, γ, λ be simple closed multicurves in S . Assume γ is an admissible representative of $[\gamma]$ in $\Sigma(\rho, \lambda)$ and $|\beta \cap \gamma| = 1$, $|\beta \cap \lambda| = 2$, and $\lambda \cap \gamma = \emptyset$. Then*

$$Gr_{T_{\beta}^{2m}(\gamma)}(\Sigma(\rho, \lambda)) = Gr_{T_{\beta}^k(\gamma)}(\Sigma(\rho, T_{\beta}^l(\lambda))).$$

Proof : Assume $m > 0$. Lemma 11.4 implies $T_{\beta}^m(\gamma)$ has a right-spiraling admissible representative in $\Sigma(\rho, \lambda)$. Then $T^k(\gamma)$ is right-spiraling in $\Sigma(\rho, T_{\beta}^{-l}(\lambda))$ and this implies that $T^k(\gamma)$ has a left-spiraling admissible representative in $\Sigma(\rho, T_{\beta}^l(\lambda))$.

Let $Gr_{T_{\beta}^{2m}(\gamma)}(\Sigma(\rho, \lambda)) = \Sigma(\rho, \lambda_1)$ and $Gr_{T_{\beta}^k(\gamma)}(\Sigma(\rho, T_{\beta}^l(\lambda))) = \Sigma(\rho, \lambda_2)$. Theorem 10.3 implies

$$\lambda_1 = [\lambda, 2T_{\beta}^{2m}(\gamma)]_{\flat} \text{ and } \lambda_2 = [T_{\beta}^l(\lambda), 2T_{\beta}^k(\gamma)]_{\#}.$$

Then Lemma 11.6 implies

$$\lambda_2 = [\lambda, 2T_{\beta}^{k+l}(\gamma)]_{\flat}.$$

A symmetric proof works for $m < 0$.

□

Corollary 11.8 *Any two real Schottky projective structures differing by an iterated elementary move can be connected in an infinite number of ways by grafting.*

Proof : Assume $\Sigma(\rho, \lambda_1)$ and $\Sigma(\rho, \lambda_2)$ are projective structures on S and $\lambda_1 = T_{\beta}^k(\lambda_2)$. Then for every $m \in \mathbb{Z}$ there is an $l \in \mathbb{Z}$ such that $m = k + l$. By Theorem 11.7 each pair (m, l) gives a different way to connect the two structures by grafting. □

Corollary 11.9 *There are an infinite number of standard projective structures that can be connected by grafting.*

Proof : Given a standard projective structure $\Sigma(\rho, \lambda)$, let β be a meridian of S . Since β develops to a closed curve in $\widehat{\mathbb{C}}$ twists about β correspond to twisting in $\widehat{\mathbb{C}}$ as in Figure 5. So for every $l \in \mathbb{Z}$ the structure $\Sigma(\rho, T_\beta^l(\lambda))$ is a standard projective structure. By Theorem 11.7 these two structures can be connected by grafting. \square

Next we prove a statement relating grafting and twists about two different meridians. Let $T_j^i = T_{\beta_j}^i$ where T^i is i Dehn twists about β_j , a meridian in S .

Corollary 11.10 *Let k, l, m be integers such that $k + l = m$ and $l \cdot k \geq 0$ and $l < k$. Let $\beta_1, \beta_2, \gamma, \lambda$ be simple closed multicurves in S and assume γ is admissible in $\Sigma(\rho, \lambda)$. Assume $|\beta_i \cap \gamma| = 1$, $|\beta_i \cap \lambda| = 2$, and $\lambda \cap \gamma = \emptyset$. Then*

$$Gr_{T_1^k T_2^l(\gamma)}(\Sigma(\rho, \lambda)) = Gr_{T_2^l(\gamma)}(\Sigma(\rho, T_1^k(\lambda))).$$

Proof : Assume first that l and k are both positive. By Lemma 11.4 $T_1^k T_2^l(\gamma)$ has an admissible right-spiraling representative in $\Sigma(\rho, \lambda)$. Since $l < k$ it also follows from Lemma 11.4 that $T_2^l(\gamma)$ is left-spiraling relative to $T_1^k(\lambda)$. Then

$$Gr_{T_1^k T_2^l(\gamma)}(\Sigma(\rho, \lambda)) = Gr_{T_2^l(\gamma)}(\Sigma(\rho, T_1^k(\lambda)))$$

by Theorem 11.7.

If $l \cdot k = 0$ and $k = 0$ then $T_1^k(\delta) = \delta$ for all closed curves δ in S . Then $T_1^k T_2^l(\gamma) = T_2^l(\gamma)$ is left-spiraling by Lemma 11.4. Then

$$Gr_{T_1^k T_2^l(\gamma)}(\Sigma(\rho, \lambda)) = Gr_{T_2^l(\gamma)}(\Sigma(\rho, \lambda))$$

by Theorem 11.7. A similar proof works for the case that $l = 0$. \square

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